G-TUTTE POLYNOMIAL VIA ALGEBRAIC COMBINATORICS, TOPOLOGY AND ENUMERATION

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This is based on the joint work with Ye Liu and Masahiko Yoshinaga

ABSTRACT. We introduce and study the notion of the G-Tutte polynomial for a list $\mathcal{A}$ of elements in a finitely generated abelian group Γ and an abelian group G, which is defined by counting the number of homomorphisms from associated finite abelian groups to G. The G-Tutte polynomial is a common generalization of the (arithmetic) Tutte polynomial for realizable (arithmetic) matroids, the characteristic quasi-polynomial for integral arrangements, Brändén-Moci's arithmetic version of the partition function of an abelian group-valued Potts model, and the modified Tutte-Krushkal-Renhardy polynomial for a finite CW-complex. As in the classical case, G-Tutte polynomials carry topological and enumerative information (e.g., the Euler characteristic, point counting and the Poincaré polynomial) of abelian Lie group arrangements. We also discuss differences between the arithmetic Tutte and the G-Tutte polynomials related to the axioms for arithmetic matroids and the (non-)positivity of coefficients.

1. INTRODUCTION

The Tutte polynomial is one of the most important invariants of a graph. The significance of the Tutte polynomial is that it has several important specializations, including chromatic polynomials, partition functions of Potts models ([28]), and Jones polynomials for alternating links ([30]). Another noteworthy aspect of the Tutte polynomial is that it depends only on the (graphical) matroid structure, and thus one can define the Tutte polynomial for a matroid. Matroids and (specializations of) Tutte polynomials play a role in several diverse areas of mathematics ([27, 31]).

Matroids and Tutte polynomials are particularly important in the study of hyperplane arrangements ([26]), because the Tutte polynomial and one of its specializations, the characteristic polynomial, carry enumerative and topological information about the arrangement. For instance, the number of points over a finite field, the number of chambers for a real arrangement and the Betti numbers for a complex arrangement are all obtained from the characteristic polynomial.

In the context of hyperplane arrangements, matroids are considered to be the data that encode the pattern of intersecting hyperplanes. It should be noted that the isomorphism class of a subspace is determined by its dimension (or codimension), or equivalently, by the rank function in matroid theory. This is the reason that matroids are extremely powerful in the study of hyperplane arrangements.

It is natural to consider arrangements of subsets of other types. There have been many attempts to consider arrangements of submanifolds inside a manifold. Recently, arrangements of subtori in a torus, or so-called toric arrangements, have received considerable attention ([12]), which has origin in the study of the moduli space of curves ([23]) and regular semisimple elements in an algebraic group ([21]).

However, beyond linear subspaces, the notion of rank is no longer sufficient to determine the isomorphism class of intersections of an arrangement. We need additional structure to describe intersection patterns combinatorially.

The notions of arithmetic Tutte polynomials and arithmetic matroids invented by Moci and collaborators ([24, 11, 8, 15]) are particularly useful for studying toric arrangements. As in the case of hyperplane arrangements, arithmetic Tutte polynomials carry enumerative and topological information about toric arrangements. It is generally difficult to explicitly compute the arithmetic Tutte polynomial. Arithmetic Tutte polynomials for classical root systems were computed by Ardila, Castillo and Henley ([1]).

Key words and phrases. Tutte polynomial, characteristic quasi-polynomial, Poincaré polynomial, arithmetic matroids.

Another (quasi-)polynomial invariant for a hyperplane arrangement defined over integers, the characteristic quasi-polynomial introduced by Kamiya, Takemura and Terao [18], is a refinement of the characteristic polynomial of an arrangement. The notion of the characteristic quasi-polynomial is closely related to Ehrhart theory on counting lattice points, and has increased in combinatorial importance recently. The characteristic quasi-polynomial for root systems was essentially computed by Suter [29] (see also [19]). By comparing the computations of Suter with those of Ardila, Castillo and Henley, it has been observed that the last constituent of the characteristic quasi-polynomial is a specialization of the arithmetic Tutte polynomial.

The purpose of this paper is to introduce and study a new class of polynomial invariant that forms a common generalization of the Tutte, arithmetic Tutte and characteristic quasi-polynomials, among others. The key observation to unify the above "Tutte-like polynomials" is that they are all defined by means of counting homomorphisms between certain abelian groups (this formulation appeared in [8, §7]). This observation has prompted us to introduce the notion of the *G-Tutte polynomial* $T^G_{\mathcal{A}}(x, y)$ for a list of elements $\mathcal{A}$ in a finitely generated abelian group Γ and an abelian group G with a certain finiteness assumption on the torsion elements (see §3.2 for details). We mainly consider abelian Lie groups G of the form

$$G = F \times (S^1)^p \times \mathbb{R}^q,$$

where F is a finite abelian group and $p, q \geq 0$. When the group G is $\mathbb{C}$, $\mathbb{C}^\times$, or the finite cyclic group $\mathbb{Z}/k\mathbb{Z}$, the G-Tutte polynomial is precisely the Tutte polynomial, the arithmetic Tutte polynomial, or a constituent of the characteristic quasi-polynomial, respectively. We will see that many known properties (deletion-contraction formula, Euler characteristic of the complement, point counting, Poincaré polynomial, convolution formula) for (arithmetic) Tutte polynomials are shared by G-Tutte polynomials. (See [14] for another attempt to generalize arithmetic Tutte polynomials.)

The organization of this paper is as follows.

§2 gives a summary of background material. We recall definitions of the Tutte polynomial $T_{\mathcal{A}}(x, y)$, arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$ and the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$ for a given list of elements $\mathcal{A}$ in $\Gamma = \mathbb{Z}^\ell$.

In §3, we study the problem from an algebraic combinatorial approach, which is a source of our main motivation. We define arrangements $\mathcal{A}(G)$ of subgroups in $\mathrm{Hom}(\Gamma, G)$ and its complements $\mathcal{M}(\mathcal{A}; \Gamma, G)$ for arbitrary abelian group G. We see that the set-theoretic deletion-contraction formula holds. In §3.2, the G-Tutte polynomial $T^G_{\mathcal{A}}(x, y)$ is defined using the number of homomorphisms of certain finite abelian groups to G (the G-multiplicities). If $G = S^1$ or $\mathbb{C}^\times$, then the G-multiplicities satisfy the five axioms of arithmetic matroids given in [11]. A natural question to ask is whether the G-multiplicities satisfy these axioms for general groups G. In §3.4, we show that four of the five axioms are satisfied by the G-multiplicities. We also prove that the G-multiplicity function satisfies another important formula, the so-called convolution formula, which has been a formula of interest recently [3, 13].

§4 contains an application of the G-Tutte polynomials via algebraic topology where we prove a formula that expresses the Poincaré polynomial of $\mathcal{M}(\mathcal{A}; \Gamma, G)$ in terms of G-characteristic polynomials under the assumption that G is a non-compact abelian Lie group with finitely many connected components. This formula covers several classical results, including hyperplane arrangements (Orlik-Solomon [25] and Zaslavsky [34]), subspace arrangements (Goresky-MacPherson [16], Björner [6]) and toric arrangements (De Concini-Procesi [12], Moci [24]).

In §5, we will be concerned with an enumerative point counting problem. We show that the Euler characteristic $e(\mathcal{M}(\mathcal{A}; \Gamma, G))$ of the complement can be computed as a special value of the G-Tutte polynomial (or G-characteristic polynomial) when G is an abelian Lie group with finitely many components. As a special case, when G is finite, we obtain a formula that counts the cardinality $\#\mathcal{M}(\mathcal{A}; \Gamma, G)$. The equality between the arithmetic characteristic polynomial and the last constituent of the characteristic quasi-polynomial is also proved. In §5.3 we compute the Poincaré polynomial for toric arrangements associated with root systems (considering positive roots to be a list in the root lattice).

This report is an extended abstract of the preprint [22], to which the interested reader is suggested to refer for many details and for all the proofs, which are omitted here.

2. Background

2.1. **(Arithmetic) Tutte polynomials.** Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}^\ell$ be a list of integer vectors, let $\alpha_i = (a_{i1}, \ldots, a_{i\ell})$. We may consider α_i to be a linear form defined by

$$\alpha_i(x_1, \ldots, x_\ell) = a_{i1}x_1 + \cdots + a_{i\ell}x_\ell.$$

A sublist $\mathcal{S} \subset \mathcal{A}$ determines a homomorphism $\alpha_{\mathcal{S}} : \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^{\#\mathcal{S}}$. Let G be an abelian group. Define

$$H_{\alpha_i,G} := \mathrm{Ker}(\alpha_i \otimes G : G^\ell \longrightarrow G) \leq G^\ell.$$

The list $\mathcal{A}$ determines an arrangement $\mathcal{A}(G) = \{H_{\alpha,G} \mid \alpha \in \mathcal{A}\}$ of subgroups in G^ℓ. Denote their complement by

$$\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, G) := G^\ell \smallsetminus \bigcup_{\alpha_i \in \mathcal{A}} H_{\alpha_i,G}.$$

The arrangement $\mathcal{A}(G)$ of subgroups and its complement $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, G)$ are important objects of study in many contexts. We list some of them below.

(i) When G is the additive group of a field (e.g., $G = \mathbb{C}, \mathbb{R}, \mathbb{F}_q$), $\mathcal{A}(G)$ is the associated hyperplane arrangement ([26]).
(ii) When $G = \mathbb{R}^c$ with $c > 0$, $\mathcal{A}(G)$ is called the c-plexification of $\mathcal{A}$ (see [6, §5.2]).
(iii) When G is $\mathbb{C}^\times$ or S^1, $\mathcal{A}(G)$ is called a toric arrangement.
(iv) When $G = S^1 \times S^1$ (viewed as an elliptic curve), $\mathcal{A}(G)$ is called an elliptic (or abelian) arrangement. ([5]).
(v) When G is a finite cyclic group $\mathbb{Z}/q\mathbb{Z}$, $\mathcal{A}(G)$ is related to the characteristic quasi-polynomial studied in [18, 19] (see 2.2). There is also an important connection with Ehrhart theory and enumerative problems ([7, 32, 33]).

To define the arithmetic Tutte polynomial, we need further notation. The linear map $\alpha_{\mathcal{S}}$ is expressed by the matrix $M_{\mathcal{S}} = (a_{ij})_{i\in\mathcal{S}, 1\leq j\leq \ell}$ of size $\#\mathcal{S} \times \ell$. Denote by $r_{\mathcal{S}}$ the rank of $M_{\mathcal{S}}$. Suppose that $d_{\mathcal{S},i}$ with $1 \leq i \leq r_{\mathcal{S}}$, $0 < d_{\mathcal{S},i}$ divides $d_{\mathcal{S},i+1}$ are the invariant factors of $M_{\mathcal{S}}$. The Tutte polynomial $T_{\mathcal{A}}(x, y)$ and the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$ of $\mathcal{A}$ are defined as follows ([24, 8]).

$$T_{\mathcal{A}}(x, y) = \sum_{\mathcal{S}\subset\mathcal{A}} (x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}},$$
$$T^{\mathrm{arith}}_{\mathcal{A}}(x, y) = \sum_{\mathcal{S}\subset\mathcal{A}} m(\mathcal{S})(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}},$$

where $m(\mathcal{S}) = \prod_{i=1}^{r_{\mathcal{S}}} d_{\mathcal{S},i}$.

These polynomials encode combinatorial and topological information about the arrangements. For instance, the characteristic polynomial of the ranked poset of flats of the hyperplane arrangement is $\chi_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} t^{\ell-r_{\mathcal{A}}} T_{\mathcal{A}}(1-t, 0)$, and the Poincaré polynomial of $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{R}^c)$ is ([16, 6])

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{R}^c)}(t) = t^{r_{\mathcal{A}}\cdot(c-1)} \cdot T_{\mathcal{A}}\left(\frac{1+t}{t^{c-1}}, 0\right). \tag{2.1}$$

Note that the special cases $c = 1$ and $c = 2$ reduce to the famous formulas given by Zaslavsky [34] and Orlik-Solomon [25], respectively. Similarly, as proved by De Concini-Procesi [12] and Moci [24], the characteristic polynomial of the layers (connected components of intersections) of the corresponding toric arrangement is $\chi^{\mathrm{arith}}_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} t^{\ell-r_{\mathcal{A}}} T^{\mathrm{arith}}_{\mathcal{A}}(1-t, 0)$, and the Poincaré polynomial of $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{C}^\times)$ is

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{C}^\times)}(t) = (1+t)^{\ell-r_{\mathcal{A}}} \cdot t^{r_{\mathcal{A}}} \cdot T^{\mathrm{arith}}_{\mathcal{A}}\left(\frac{1+2t}{t}, 0\right). \tag{2.2}$$

The cohomology ring structure of $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{C}^\times)$ was recently described by Callegaro-Delucchi [9].

2.2. **Characteristic quasi-polynomials.** Kamiya, Takemura and Terao [18] proved that $\#\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}/q\mathbb{Z})$ is a quasi-polynomial in q ($q\in\mathbb{Z}_{>0}$), denoted by $\chi^{\text{quasi}}_{\mathcal{A}}(q)$, with period

$$\rho_{\mathcal{A}} := \operatorname{lcm}(d_{\mathcal{S},r_{\mathcal{S}}} \mid \mathcal{S}\subset\mathcal{A}).$$

More precisely, there exist polynomials $f_1(t), f_2(t),\cdots, f_{\rho_{\mathcal{A}}}(t)\in\mathbb{Z}[t]$ such that for any positive integer q,

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) := \#\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}/q\mathbb{Z}) = f_k(q),$$

where $k\equiv q \mod \rho_{\mathcal{A}}$. The polynomial $f_k(t)$ is called the k-constituent. They also proved that $f_k(t)=f_m(t)$ if $\gcd(k,\rho_{\mathcal{A}})=\gcd(m,\rho_{\mathcal{A}})$. Furthermore, the 1-constituent $f_1(t)$ (and more generally, $f_k(t)$ with $\gcd(k,\rho_{\mathcal{A}})=1$) is known to be equal to the characteristic polynomial $\chi_{\mathcal{A}}(t)$ ([2]).

We will show that the most degenerate constituent $f_{\rho_{\mathcal{A}}}(t)$ is obtained as a specialization of the arithmetic Tutte polynomial, and that the other constituents can also be described in terms of the G-Tutte polynomials introduced later (Theorem 5.3, Corollary 5.4).

3. Algebraic Combinatorics

Throughout the paper, the term **list** refers to synonym of multiset. Let Γ be a finitely generated abelian group, $\mathcal{A}\subset\Gamma$ a list of finitely many elements, and G an arbitrary abelian group.

3.1. **Arrangements over abelian groups.** Let us denote the subgroup of torsion elements of Γ by $\Gamma_{\text{tor}}\subset\Gamma$, and the rank of Γ by r_Γ. More generally, for a sublist $\mathcal{S}\subset\Gamma$, denote the rank of the subgroup $\langle\mathcal{S}\rangle\subset\Gamma$ generated by $\mathcal{S}$ by

$$r_{\mathcal{S}} = \operatorname{rank}\langle\mathcal{S}\rangle.$$

We now define the "arrangement" associated with a list $\mathcal{A}$ over an arbitrary abelian group G. The total space is the abelian group

$$\operatorname{Hom}(\Gamma,G) = \{\varphi:\Gamma\longrightarrow G \mid \varphi \text{ is a homomorphism}\}$$

of all homomorphisms from Γ to G. For each $\alpha\in\Gamma$, define

$$H_{\alpha,G} := \{\varphi\in\operatorname{Hom}(\Gamma,G)\mid\varphi(\alpha)=0\}.$$

The collection of subgroups $\mathcal{A}(G)=\{H_{\alpha,G}\mid\alpha\in\mathcal{A}\}$ is called the G-plexification of $\mathcal{A}$. Denote the complement of $\mathcal{A}(G)$ by

$$\mathcal{M}(\mathcal{A};\Gamma,G) := \operatorname{Hom}(\Gamma,G)\smallsetminus\bigcup_{\alpha\in\mathcal{A}} H_{\alpha,G}.$$

Fix $\alpha\in\mathcal{A}$. Denote $\mathcal{A}' := \mathcal{A}\smallsetminus\{\alpha\}$ as a list of elements in the same group $\Gamma' := \Gamma$. Set $\Gamma'' := \Gamma/\langle\alpha\rangle$, and $\mathcal{A}'' := \mathcal{A}/\{\alpha\} = \{\overline{\alpha'}\mid\alpha'\in\mathcal{A}'\}\subseteq\Gamma''$, the contraction of $\mathcal{A}$ to $\{\alpha\}$. The group $\operatorname{Hom}(\Gamma'',G)$ can be identified with

$$H_{\alpha,G} = \{\varphi\in\operatorname{Hom}(\Gamma,G)\mid\varphi(\alpha)=0\}.$$

Proposition 3.1 (Deletion-Contraction formula)**.**

$$\mathcal{M}(\mathcal{A}';\Gamma',G) = \mathcal{M}(\mathcal{A};\Gamma,G)\sqcup\mathcal{M}(\mathcal{A}'';\Gamma'',G).$$

3.2. G**-Tutte polynomials.**

Definition 3.2. An abelian group G is said to be *torsion-wise finite* if the subgroup of d-torsion points $G[d] := \{x\in G\mid d\cdot x=0\}$ is finite for all $d\in\mathbb{Z}_{>0}$.

Convention: In the remaining of this paper, we assume that G is always a torsion-wise finite group.

Definition 3.3. The G-*multiplicity* $m(\mathcal{S};G)\in\mathbb{Z}_{>0}$ for each $\mathcal{S}\subseteq\Gamma$ is defined by

$$m(\mathcal{S};G) := \#\operatorname{Hom}\left((\Gamma/\langle\mathcal{S}\rangle)_{\text{tor}},G\right).$$

Definition 3.4.

(1) The *multivariate G-Tutte polynomial* $Z^G_{\mathcal{A}}(q, \boldsymbol{v})$ of $\mathcal{A}$ is defined by
$$Z^G_{\mathcal{A}}(q, \boldsymbol{v}) = Z^G_{\mathcal{A}}(q, v_1, \ldots, v_n) := \sum_{\mathcal{S}\subseteq\mathcal{A}} m(\mathcal{S}; G) q^{-r_{\mathcal{S}}} \prod_{\alpha_i\in\mathcal{S}} v_i.$$
(2) The *G-Tutte polynomial* $T^G_{\mathcal{A}}(x, y)$ of $\mathcal{A}$ is defined by
$$T^G_{\mathcal{A}}(x, y) := \sum_{\mathcal{S}\subseteq\mathcal{A}} m(\mathcal{S}; G)(x - 1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y - 1)^{\#\mathcal{S}-r_{\mathcal{S}}}.$$
(3) The *G-characteristic polynomial* $\chi^G_{\mathcal{A}}(t)$ of $\mathcal{A}$ is defined by
$$\chi^G_{\mathcal{A}}(t) := \sum_{\mathcal{S}\subseteq\mathcal{A}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; G) \cdot t^{r_\Gamma - r_{\mathcal{S}}}.$$

These three polynomials are related by the following formulas:
$$T^G_{\mathcal{A}}(x, y) = (x - 1)^{r_{\mathcal{A}}} \cdot Z^G_{\mathcal{A}}((x - 1)(y - 1), y - 1, \ldots, y - 1),$$
$$\chi^G_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T^G_{\mathcal{A}}(1 - t, 0).$$

Recall that $\alpha \in \mathcal{A}$ is called a loop (resp. coloop) if $\alpha \in \Gamma_{\mathrm{tor}}$ (resp. $r_{\mathcal{A}} = r_{\mathcal{A}\smallsetminus\{\alpha\}} + 1$). An element α that is neither a loop nor a coloop is called proper ([11, §4.4]).

Lemma 3.5. *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple associated with $\alpha_i \in \mathcal{A}$. Then*
$$Z^G_{\mathcal{A}}(q, \boldsymbol{v}) = \begin{cases} Z^G_{\mathcal{A}'}(q, \boldsymbol{v}) + v_i \cdot Z^G_{\mathcal{A}''}(q, \boldsymbol{v}), & \text{if } \alpha_i \text{ is a loop,} \\ Z^G_{\mathcal{A}'}(q, \boldsymbol{v}) + v_i \cdot q^{-1} \cdot Z^G_{\mathcal{A}''}(q, \boldsymbol{v}), & \text{otherwise.} \end{cases}$$

Corollary 3.6. *The G-Tutte polynomials satisfy*
$$T^G_{\mathcal{A}}(x, y) = \begin{cases} T^G_{\mathcal{A}'}(x, y) + (y - 1)T^G_{\mathcal{A}''}(x, y), & \text{if } \alpha_i \text{ is a loop,} \\ (x - 1)T^G_{\mathcal{A}'}(x, y) + T^G_{\mathcal{A}''}(x, y), & \text{if } \alpha_i \text{ is a coloop,} \\ T^G_{\mathcal{A}'}(x, y) + T^G_{\mathcal{A}''}(x, y), & \text{if } \alpha_i \text{ is proper.} \end{cases}$$

Corollary 3.7. *The G-characteristic polynomials satisfy*
$$\chi^G_{\mathcal{A}}(t) = \chi^G_{\mathcal{A}'}(t) - \chi^G_{\mathcal{A}''}(t).$$

3.3. **Specializations.** The G-Tutte polynomial has several specializations. We choose to mention some of important ones.

Proposition 3.8. *Let $\mathcal{A}$ be a list in the free abelian group $\Gamma = \mathbb{Z}^\ell$.*

(1) *Suppose that G is a torsion-free abelian group. Then $T^G_{\mathcal{A}}(x, y) = T_{\mathcal{A}}(x, y)$ and $\chi^G_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t)$.*
(2) *Suppose $G = S^1$ or $\mathbb{C}^\times$. Then $T^G_{\mathcal{A}}(x, y) = T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$ and $\chi^G_{\mathcal{A}}(t) = \chi^{\mathrm{arith}}_{\mathcal{A}}(t)$.*

The arithmetic Tutte polynomial can also be obtained as another specialization of the G-Tutte polynomial. Suppose that $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \simeq \bigoplus_{i=1}^{k_{\mathcal{S}}} \mathbb{Z}/d_{\mathcal{S},i}\mathbb{Z}$, where $k_{\mathcal{S}} \geq 0$ and $d_{\mathcal{S},i}|d_{\mathcal{S},i+1}$. Define $\rho_{\mathcal{A}}$ by
$$\rho_{\mathcal{A}} := \mathrm{lcm}(d_{\mathcal{S},k_{\mathcal{S}}} \mid \mathcal{S} \subset \mathcal{A}).$$

Proposition 3.9. $T^{\mathbb{Z}/\rho_{\mathcal{A}}\mathbb{Z}}_{\mathcal{A}}(x, y) = T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$.

3.4. **Relationship with arithmetic matroids.**

Theorem 3.10. *The G-multiplicities satisfy the following four properties (we borrow the numbering from* [11, §2.3]*).*

(1) *If $\mathcal{S} \subset \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}}$, then $m(\mathcal{S} \cup \{\alpha\}; G)$ divides $m(\mathcal{S}; G)$.*
(2) *If $\mathcal{S} \subset \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}} + 1$, then $m(\mathcal{S}; G)$ divides $m(\mathcal{S} \cup \{\alpha\}; G)$.*
(4) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $r_{\mathcal{S}} = r_{\mathcal{T}}$, then*
$$\rho_{\mathcal{T}}(\mathcal{S}; G) := \sum_{\mathcal{S}\subset\mathcal{B}\subset\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} m(\mathcal{B}; G) \geq 0.$$

(5) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $r_{\mathcal{T}} = r_{\mathcal{S}} + \#(\mathcal{T} \setminus \mathcal{S})$, then*

$$\rho^*_{\mathcal{T}}(\mathcal{S}; G) := \sum_{\mathcal{S} \subset \mathcal{B} \subset \mathcal{T}} (-1)^{\#\mathcal{T} - \#\mathcal{B}} m(\mathcal{B}; G) \geq 0.$$

Additionally, if G is a (torsion-wise finite) divisible abelian group, that is, the multiplication-by-k map $k : G \longrightarrow G$ is surjective for any positive integer k, then the G-multiplicities satisfy the following.

(3) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $\mathcal{T}$ is a disjoint union $\mathcal{T} = \mathcal{S} \sqcup \mathcal{B} \sqcup \mathcal{C}$ such that for all $\mathcal{S} \subset \mathcal{R} \subset \mathcal{T}$, we have $r_{\mathcal{R}} = r_{\mathcal{S}} + \#(\mathcal{R} \cap \mathcal{B})$, then*

$$m(\mathcal{S}; G) \cdot m(\mathcal{T}; G) = m(\mathcal{S} \sqcup \mathcal{B}; G) \cdot m(\mathcal{S} \sqcup \mathcal{C}; G).$$

Theorem 3.11. *Let G be a torsion-wise finite divisible abelian group. Then the coefficients of the G-Tutte polynomial $T^G_{\mathcal{A}}(x, y)$ are positive integers.*

Theorem 3.12 (Convolution formula)**.** *Let $\mathcal{A} \subset \Gamma$ be a list in a finitely generated group Γ, and let G_1 and G_2 be torsion-wise finite groups. Then*

$$T^{G_1 \times G_2}_{\mathcal{A}}(x, y) = \sum_{\mathcal{B} \subset \mathcal{A}} T^{G_1}_{\mathcal{B}}(0, y) \cdot T^{G_2}_{\mathcal{A}/\mathcal{B}}(x, 0).$$

4. Algebraic Topology

4.1. **Torus cycles.** We introduce a special class of homology cycles in $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$, called torus cycles, which are lifts of cycles in a compact torus. Let $G = F \times (S^1)^p \times \mathbb{R}^q$, where F a finite abelian group. Write $G_c = F \times (S^1)^p$ (compact part) and $V = \mathbb{R}^q$ (non-compact part). Let Γ be a finitely generated abelian group. Fix a decomposition $\Gamma = \Gamma_{\text{tor}} \oplus \Gamma_{\text{free}}$, where $\Gamma_{\text{free}} \simeq \mathbb{Z}^{r_\Gamma}$. Then

$$\operatorname{Hom}(\Gamma, G) \simeq \operatorname{Hom}(\Gamma, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, V). \tag{4.1}$$

(Note that $\operatorname{Hom}(\Gamma_{\text{tor}}, V) = 0$). We can decompose this further as follows:

$$\operatorname{Hom}(\Gamma, G) \simeq \operatorname{Hom}(\Gamma_{\text{tor}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, V). \tag{4.2}$$

Let $\alpha = (\beta, \eta) \in \Gamma_{\text{tor}} \oplus \Gamma_{\text{free}}$. According to decomposition (4.1),

$$H_{\alpha,G} = H_{\alpha,G_c} \times H_{\eta,V},$$

where $H_{\alpha,G_c} \subset \operatorname{Hom}(\Gamma, G_c)$ and $H_{\eta,V} \subset \operatorname{Hom}(\Gamma_{\text{free}}, V)$. If $\alpha \in \Gamma_{\text{tor}}$, or equivalently $\alpha = (\beta, 0)$, then using (4.2) gives

$$H_{\alpha,G} = H_{\beta,G_c} \times \operatorname{Hom}(\Gamma_{\text{free}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, V),$$

where H_{β,G_c} is a subgroup of the finite abelian group $\operatorname{Hom}(\Gamma_{\text{tor}}, G_c)$. In this case, $H_{\alpha,G}$ is a collection of connected components of $\operatorname{Hom}(\Gamma, G)$. If $\mathcal{A} \subset \Gamma_{\text{tor}} \subset \Gamma$, then

$$\mathcal{M}(\mathcal{A}; \Gamma, G) = \mathcal{M}(\mathcal{A}; \Gamma_{\text{tor}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, V). \tag{4.3}$$

Therefore, $\mathcal{M}(\mathcal{A}; \Gamma, G)$ is also a collection of some of connected components of $\operatorname{Hom}(\Gamma, G)$.

Let $\mathcal{A} \subset \Gamma$ be a list of elements. Define $\mathcal{A}_{\text{tor}} := \mathcal{A} \cap \Gamma_{\text{tor}}$. Consider the following diagram:

$$\begin{tikzcd} \mathcal{M}(\mathcal{A}; \Gamma, G) \arrow[r, "\subset"] & \mathcal{M}(\mathcal{A}_{\text{tor}}; \Gamma, G) \arrow[r, "\subset"] \arrow[d] & \operatorname{Hom}(\Gamma, G) \ni (f, t, v) \arrow[d, "\pi"] \\ & \mathcal{M}(\mathcal{A}_{\text{tor}}; \Gamma, G_c) \arrow[r, "\subset"] & \operatorname{Hom}(\Gamma, G_c) \ni (f, t), \end{tikzcd} \tag{4.4}$$

where $\pi : \operatorname{Hom}(\Gamma, G) \longrightarrow \operatorname{Hom}(\Gamma, G_c)$ is the projection defined by $\pi(f, t, v) = (f, t)$ for $(f, t, v) \in \operatorname{Hom}(\Gamma_{\text{tor}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, G_c) \times \operatorname{Hom}(\Gamma_{\text{free}}, V) \simeq \operatorname{Hom}(\Gamma, G)$.

Now assume that $q > 0$. The fiber of the projection π is isomorphic to $\operatorname{Hom}(\Gamma, V) \simeq V^{r_\Gamma} \simeq \mathbb{R}^{q \cdot r_\Gamma}$. Hence

$$\mathcal{M}(\mathcal{A} \setminus \mathcal{A}_{\text{tor}}; \Gamma, V) = \operatorname{Hom}(\Gamma, V) \setminus \bigcup_{\alpha \in \mathcal{A} \setminus \mathcal{A}_{\text{tor}}} H_{\alpha,V}$$

is non-empty. Fix an element $v_0 \in \mathcal{M}(\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}; \Gamma, V)$. For a given $(f, t) \in \mathrm{Hom}(\Gamma, G_c)$, define $i_{v_0}(f, t) := (f, t, v_0)$. This induces a map

$$i_{v_0} : \mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G_c) \longrightarrow \mathcal{M}(\mathcal{A}; \Gamma, G),$$

which is a section of the projection $\pi|_{\mathcal{M}(\mathcal{A};\Gamma,G)} : \mathcal{M}(\mathcal{A}; \Gamma, G) \longrightarrow \mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G_c)$ in (4.4).

Definition 4.1. Assume that $q > 0$. A cycle $\gamma \in H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is said to be a *torus cycle* if there exist a connected component $T \subset \mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G_c)$, a cycle $\widetilde{\gamma} \in H_*(T, \mathbb{Z}) \subset H_*(\mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G_c), \mathbb{Z})$ and $v_0 \in \mathcal{M}(\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}; \Gamma, V)$ such that

$$\gamma = (i_{v_0})_*(\widetilde{\gamma}).$$

The subgroup of $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ generated by torus cycles is denoted by $H_*^{\mathrm{torus}}(\mathcal{A}(G))$.

4.2. **Meridian cycles.** The torus cycles introduced in the previous section are not enough to generate the homology group $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. We also need to consider meridians of $H_{\alpha,G}$ to generate $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. Let us first recall the notion of layers. A layer of $\mathcal{A}(G)$ is a connected component of a non-empty intersection of elements of $\mathcal{A}(G)$. Let $\mathcal{S} \subset \mathcal{A}$. Every connected component of $H_{\mathcal{S},G} := \bigcap_{\alpha\in\mathcal{S}} H_{\alpha,G}$ is isomorphic to $\left((S^1)^p \times \mathbb{R}^q\right)^{r_\Gamma - r_{\mathcal{S}}}$. We sometimes call the number $r_{\mathcal{S}}$ the rank of the layer. Since $H_{\emptyset,G} = \mathrm{Hom}(\Gamma, G)$, a connected component of $\mathrm{Hom}(\Gamma, G)$ is a layer of rank 0. Similarly, a connected component of $H_{\alpha,G}$ for $\alpha \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}$ is a layer of rank 1.

Let L be a layer. Denote the set of α such that $H_{\alpha,G}$ contains L by $\mathcal{A}_L := \{\alpha \in \mathcal{A} \mid L \subset H_{\alpha,G}\}$, and the contraction by $\mathcal{A}^L := \mathcal{A}/\mathcal{A}_L$. Note that L can be considered to be a rank 0 layer of $\mathcal{A}^L(G)$. Define

$$\begin{aligned}\mathcal{M}^L(\mathcal{A}) :=& L \smallsetminus \bigcup_{H_{\alpha,G} \not\supset L} H_{\alpha,G} \\ =& L \cap \mathcal{M}(\mathcal{A}^L; \Gamma/\langle \mathcal{A}_L \rangle, G).\end{aligned}$$

Let $L_1 \subset \mathrm{Hom}(\Gamma, G)$ be a rank 1 layer of $\mathcal{A}(G)$, and let L_0 be the rank 0 layer that contains L_1. We wish to define the meridian homomorphism

$$\mu^\varepsilon_{L_0/L_1} : H_*(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z}),$$

where $g = \dim G = p + q > 0$ and $\varepsilon \in \{0, 1\}$.

Since the normal bundle of L_1 in L_0 is trivial, there is a tubular neighborhood U of $\mathcal{M}^{L_1}(\mathcal{A})$ in L_0 such that $U \simeq \mathcal{M}^{L_1}(\mathcal{A}) \times D^g$ with the identification $\mathcal{M}^{L_1}(\mathcal{A}) = \mathcal{M}^{L_1}(\mathcal{A}) \times \{0\}$. Then $U \cap \mathcal{M}^{L_0}(\mathcal{A}) \simeq \mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*}$, where $D^{g*} = D^g \smallsetminus \{0\}$. We denote the corresponding inclusion by $i : \mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*} \hookrightarrow M^{L_0}(\mathcal{A})$. For a given $\gamma \in H_*(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z})$, define the element $\mu^\varepsilon_{L_0/L_1}(\gamma) \in H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z})$ as follows.

(0) For $\varepsilon = 0$, let $p_0 \in D^{g*}$. Then $\gamma \times [p_0] \in H_*(\mathcal{M}^{L_1}(\mathcal{A})) \otimes H_0(D^{g*}) \subset H_*(\mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*})$, and $\mu^0_{L_0/L_1}(\gamma) := i_*(\gamma \times [p_0])$.

(1) For $\varepsilon = 1$, let $S^{g-1} \subset D^{g*}$ be a sphere of small radius. Then $\gamma \times [S^{g-1}] \in H_*(\mathcal{M}^{L_1}(\mathcal{A})) \otimes H_{g-1}(D^{g*}) \subset H_{*+g-1}(\mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*})$ (this part is essentially the Gysin homomorphism). Now define $\mu^1_{L_0/L_1}(\gamma) := i_*(\gamma \times [S^{g-1}])$.

Similarly, we can define the meridian map

$$\mu^\varepsilon_{L_j/L_{j+1}} : H_*(\mathcal{M}^{L_{j+1}}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_j}(\mathcal{A}), \mathbb{Z})$$

between layers $L_j \supset L_{j+1}$ with consecutive ranks.

Definition 4.2. A cycle $\gamma \in H_d(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is called a *meridian cycle* if there exists some $k \geq 0$ and

(a) a flag $L_0 \supset L_1 \supset \cdots \supset L_k$ of layers with $\mathrm{rank}\, L_j = j$, such that $L_0 \cap \mathcal{M}(\mathcal{A}; \Gamma, G) \neq \emptyset$ (or equivalently, $L_0 \subset \mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G)$),

(b) a sequence $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$, and

(c) a torus cycle $\tau \in H_{d-(g-1)\sum_{i=1}^k \varepsilon_i}(\mathcal{M}^{L_k}(\mathcal{A}), \mathbb{Z})$, such that

$$\gamma = \mu^{\varepsilon_1}_{L_0/L_1} \circ \mu^{\varepsilon_2}_{L_1/L_2} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau).$$

We call the minimum such k the depth of γ.

4.3. **Poincaré polynomials for non-compact groups.** Throughout this section, we assume that $G = (S^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group, $q > 0$, and $g := \dim G = p + q$. The Poincaré polynomial of G is $P_G(t) = (1+t)^p \times \#F$. For simplicity, we also set $\mathcal{M}(\mathcal{A}) := \mathcal{M}(\mathcal{A};\Gamma, G)$, $\mathcal{M}(\mathcal{A}') := \mathcal{M}(\mathcal{A}';\Gamma, G)$, and $\mathcal{M}(\mathcal{A}'') := \mathcal{M}(\mathcal{A}'';\Gamma'', G)$.

Theorem 4.3. *The following hold:*

(i) $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z})$ *is generated by meridian cycles. That is* $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z}) = H^{\mathrm{merid}}_*(\mathcal{A}(G))$, *and furthermore it is torsion free.*

(ii) *If* α *is not a loop, then* $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_*(\mathcal{M}(\mathcal{A}'), \mathbb{Z})$ *is surjective.*

(iii) *Let* $\alpha \in \mathcal{A}$. *Then*

$$P_{\mathcal{M}(\mathcal{A})}(t) = \begin{cases} P_{\mathcal{M}(\mathcal{A}')}(t) - P_{\mathcal{M}(\mathcal{A}'')}(t), & \text{if } \alpha \text{ is a loop,} \\ P_{\mathcal{M}(\mathcal{A}')}(t) + t^{g-1} \cdot P_{\mathcal{M}(\mathcal{A}'')}(t), & \text{if } \alpha \text{ is not a loop.} \end{cases}$$

Theorem 4.4.

$$\begin{aligned} P_{\mathcal{M}(\mathcal{A})}(t) &= P_G(t)^{r_\Gamma - r_\mathcal{A}} \cdot t^{r_\mathcal{A}(g-1)} \cdot T^G_\mathcal{A}\left(\frac{P_G(t)}{t^{g-1}} + 1, 0\right) \\ &= (-t^{g-1})^{r_\Gamma} \cdot \chi^G_\mathcal{A}\left(-\frac{P_G(t)}{t^{g-1}}\right). \end{aligned}$$

5. Enumeration

5.1. **Euler characteristic of the complement.** We briefly recall the notion of Euler characteristic for semialgebraic sets (see [10, 4] for further details). Every semialgebraic set X has a decomposition $X = \bigsqcup_{i=1}^N X_i$ such that each X_i is a semialgebraic subset that is semialgebraically homeomorphic to the open simplex $\sigma_{d_i} = \{(x_1, \ldots, x_{d_i}) \in \mathbb{R}^{d_i} \mid x_i > 0, \sum x_i < 1\}$ for some $d_i = \dim X_i$. The semialgebraic Euler characteristic of X is defined by

$$e_{\mathrm{semi}}(X) := \sum_{i=1}^N (-1)^{d_i}.$$

Unlike the topological Euler characteristic $e_{\mathrm{top}}(X) := \sum(-1)^i b_i(X)$, the semialgebraic Euler characteristic $e_{\mathrm{semi}}(X)$ is not homotopy invariant. However, if X is a manifold (without boundary), then $e_{\mathrm{semi}}(X)$ and $e_{\mathrm{top}}(X)$ are related by the following formula:

$$e_{\mathrm{semi}}(X) = (-1)^{\dim X} \cdot e_{\mathrm{top}}(X).$$

Here we assume that G is of the form $G = (S^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group. Such a group G can be realized as a semialgebraic set, with the group operations defined by C^∞ semialgebraic maps. Hence subsets defined by using group operations are always semialgebraic sets. The Euler characteristics of G are easily computed as

$$e_{\mathrm{semi}}(G) = \begin{cases} 0, & \text{if } p > 0, \\ (-1)^{p+q} \cdot \#F, & \text{if } p = 0, \end{cases}$$

$$e_{\mathrm{top}}(G) = \begin{cases} 0, & \text{if } p > 0, \\ \#F, & \text{if } p = 0. \end{cases}$$

Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ. The space $\mathcal{M}(\mathcal{A};\Gamma, G)$ is a semialgebraic set, and, if it is not empty, it is also a manifold (without boundary) of $\dim \mathcal{M}(\mathcal{A};\Gamma, G) = r_\Gamma \cdot \dim G$. The G-Tutte polynomial can be used to compute the Euler characteristic of $\mathcal{M}(\mathcal{A};\Gamma, G)$.

Theorem 5.1. *Let G be an abelian Lie group with finitely many connected components, and let $g = \dim G$. Then,*

$$e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G)) = \chi^G_{\mathcal{A}}(e_{\mathrm{semi}}(G)),$$

or equivalently,

$$e_{\mathrm{top}}(\mathcal{M}(\mathcal{A};\Gamma,G)) = (-1)^{g\cdot r_\Gamma}\cdot \chi^G_{\mathcal{A}}\left((-1)^g\cdot e_{\mathrm{top}}(G)\right).$$

5.2. **Point counting in complements.** In the case that G is finite, the complement $\mathcal{M}(\mathcal{A};\Gamma,G)$ is also a finite set. Every finite set can be considered as a 0-dimensional semialgebraic set whose Euler characteristic is equal to its cardinality. The following theorem immediately follows from Theorem 5.1.

Theorem 5.2. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ, and let G be a finite abelian group. Then*

$$\#\mathcal{M}(\mathcal{A};\Gamma,G) = \chi^G_{\mathcal{A}}(\#G).$$

Theorem 5.3. *(See §2.2 for notation.) Let $\mathcal{A}$ be a finite list of elements in $\Gamma = \mathbb{Z}^\ell$, and let k be a divisor of $\rho_{\mathcal{A}}$. The k-constituent $f_k(t)$ of the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$ is equal to*

$$f_k(t) = \chi^{\mathbb{Z}/k\mathbb{Z}}_{\mathcal{A}}(t).$$

Corollary 5.4. *The last constituent $f_{\rho_{\mathcal{A}}}(t)$ of the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$ is equal to both $\chi^{\mathbb{C}^\times}_{\mathcal{A}}(t)$ and $\chi^{\mathrm{arith}}_{\mathcal{A}}(t)$.*

5.3. **Examples: root systems.** Let Φ be an irreducible root system of rank ℓ, and let $\Gamma = \mathbb{Z}\cdot\Phi$ be the root lattice of Φ. Consider the list $\mathcal{A}_\Phi := \Phi^+ \subset \Gamma$ of positive roots. The characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}_\Phi}(q)$ was computed by Suter [29] and Kamiya-Takemura-Terao [19]. Using formula (2.2), or Theorem 4.4, the Poincaré polynomial for the corresponding toric arrangement is $P_{\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)}(t) = (-t)^\ell \chi^{\mathbb{C}^\times}_{\mathcal{A}_\Phi}\left(-\frac{1+t}{t}\right)$. We only show exceptional cases.

$$\begin{aligned}
P_{\mathcal{M}(\mathcal{A}_{E_6};\Gamma,\mathbb{C}^\times)}(t) =& 1 + 42t + 705t^2 + 6020t^3 + 27459t^4 + 63378t^5 + 58555t^6\\
P_{\mathcal{M}(\mathcal{A}_{E_7};\Gamma,\mathbb{C}^\times)}(t) =& 1 + 70t + 2016t^2 + 30800t^3 + 268289t^4 + 1328670t^5\\
&+ 3479734t^6 + 3842020t^7\\
P_{\mathcal{M}(\mathcal{A}_{E_8};\Gamma,\mathbb{C}^\times)}(t) =& 1 + 128t + 6888t^2 + 202496t^3 + 3539578t^4 + 37527168t^5\\
&+ 235845616t^6 + 818120000t^7 + 1313187309t^8\\
P_{\mathcal{M}(\mathcal{A}_{F_4};\Gamma,\mathbb{C}^\times)}(t) =& 1 + 28t + 286t^2 + 1260t^3 + 2153t^4\\
P_{\mathcal{M}(\mathcal{A}_{G_2};\Gamma,\mathbb{C}^\times)}(t) =& 1 + 8t + 19t^2
\end{aligned}$$

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